On resonant elliptic systems with rapidly rotating nonlinearities *

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Abstract

We study a Neumann problem for a nonlinear elliptic system. Unlike previous results in the literature of Landesman-Lazer type, our existence theorem allows rapid rotations on the nonlinear term.

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1 Introduction

We consider the Neumann problem for the elliptic system

$$\begin{cases} \Delta u + g(u) = p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $g \in C(\mathbb{R}^n, \mathbb{R}^n)$ and $p \in C(\overline{\Omega}, \mathbb{R}^n)$ has zero average, i.e.

$$\overline{p} := \frac{1}{\text{meas}(\Omega)} \int_\Omega p = 0.$$

This problem has been extensively studied. Due to its resonant structure, it is still an open problem to characterize the range of the semilinear operator $\Delta u + g(u)$, i.e. the set of all possible functions p such that (1) admits at least one weak solution. For a single equation the well-known Landesman-Lazer theorem takes the following form:

Theorem 1.1 (Landesman-Lazer) *Let $g \in C(\mathbb{R}, \mathbb{R})$ be bounded. Assume that the limits $g(\pm\infty) := \lim_{u \to \pm\infty} g(u)$ exist and satisfy either*

$$g(-\infty) < 0 < g(+\infty)$$

or

$$g(+\infty) < 0 < g(-\infty).$$

Then for each $p \in C(\overline{\Omega}, \mathbb{R})$ with $\overline{p} = 0$ problem (1) admits at least one solution.

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This theorem has been generalized in several ways. On the one hand, analogous versions have been obtained for nonlinear operators of p-Laplacian or Φ-Laplacian type. On the other hand, the assumption on the existence of limits can be relaxed. For instance, it is easy to prove that the result is still valid under the weaker condition

$$g(-u)g(u) < 0 \quad \text{for } \ u \geq R \tag{2}$$

for some large enough R. From a topological point of view, condition (2) says two different things: firstly, that g does not vanish outside a compact set; secondly, that its Brouwer degree over the interval $(-R, R)$ is different from zero when R is large. Thus, one might believe that a natural extension of the preceding result for a system of n equations could be to require that

$$g(u) \neq 0 \quad \text{for } |u| \geq R \tag{3}$$

and

$$\deg(g, B_R(0), 0) \neq 0, \tag{4}$$

where 'deg' refers to the Brouwer degree of the function $g \in C(\mathbb{R}^n, \mathbb{R}^n)$ and $B_R(0)$ is the open ball of radius R centered at the origin. For $N = 1$ this possible extension was analyzed by Ortega and Sánchez in [7], where they constructed an example showing that (3) and (4) are not sufficient to guarantee the existence of a solution. Specifically, for $n = 2$ they defined, in complex notation,

$$g_0(z) := \frac{z}{\sqrt{1 + |z|^2}} e^{i\mathrm{Re}(z)},$$

$$g(z) := g_0(z) - \gamma \quad \text{with } 0 < \gamma < 1, \tag{5}$$

and showed that problem

$$z'' + g(z) = \lambda \sin t$$

does not have a 2π-periodic solution when λ is large enough.

Already in the early seventies Nirenberg [5] proved the following generalization of the Landesman-Lazer result for systems:

Theorem 1.2 (Nirenberg) *Let $g \in C(\mathbb{R}^n, \mathbb{R}^n)$ be bounded. Assume that the radial limits*

$$g_v := \lim_{s \to +\infty} g(sv)$$

exist uniformly and $g_v \neq 0$ for every $v \in \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$. Furthermore, assume that (4) holds for R sufficiently large. Then for each $p \in C(\overline{\Omega}, \mathbb{R}^n)$ with $\overline{p} = 0$ problem (1) admits at least one solution.

As for a single equation, it is possible to replace the hypothesis on existence of limits at infinity by an interpretation of (2) for $n > 1$ which is more accurate than (3)-(4). This was done by Ruiz and Ward in [9]. The following result is adapted from their main theorem.

We write $B_r(v) := \{x \in \mathbb{R}^n : |x - v| < r\}$ and $\overline{B}_r(v)$ for its closure, and $\mathrm{co}(A)$ for the convex hull of a subset A of $\mathbb{R}^n$.

Theorem 1.3 (Ruiz-Ward) *Assume that $g \in C(\mathbb{R}^n, \mathbb{R}^n)$ is bounded and satisfies the following condition:*
For each $r > 0$ there exists $R > r$ such that

$$0 \notin \mathrm{co}(g(\overline{B}_r(v))) \qquad \text{if } \ v \in \mathbb{R}^n \ \text{and } \ |v| = R. \tag{6}$$

Then, if (4) holds, problem (1) admits at least one solution for each $p \in C(\overline{\Omega}, \mathbb{R}^n)$ with $\overline{p} = 0$.

This result was established in [9] for a system of ordinary differential equations with periodic boundary conditions although, as the authors mention, its generalization to the Neumann problem (1) in a bounded smooth domain of higher dimension is straightforward. Theorem 1.3 still holds if g is unbounded but *sublinear*, that is,

$$\frac{g(u)}{|u|} \to 0 \qquad \text{as } \ |u| \to \infty. \tag{7}$$

In a recent work [2], the result has been extended also for singular g.

The role of condition (6) becomes clear when (1) is solved by Leray-Schauder degree methods. Indeed, the key step for proving Theorem 1.3 consists in showing that, for $0 < \lambda \leq 1$, problem

$$\begin{cases} \Delta u = \lambda(p - g(u)) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

has no solution on $\partial\mathcal{U}$, where

$$\mathcal{U} := \{u \in C(\overline{\Omega}, \mathbb{R}^n) : \|u - \overline{u}\|_\infty < r, \ \ |\overline{u}| < R\}$$

for some suitable r and the corresponding R given by condition (6). An appropriate value of r is obtained after observing that, if u is satisfies (8), then

$$\|\nabla u\|_\infty \leq Q\|\Delta u\|_\infty \leq Q\left(\|p\|_\infty + \|g\|_\infty\right) \tag{9}$$

for some constant Q, independent of p and g (but depending on Ω). This yields the a priori bound $\|u - \overline{u}\|_\infty < r$ for r large enough. Next, if we assume that $|\overline{u}| = R$, we obtain a contradiction as follows: since the convex hull of $g(\overline{B}_r(\overline{u}))$ is compact, the geometric version of the Hahn-Banach theorem asserts that there exists a hyperplane H passing through the origin such that $g(\overline{B}_r(\overline{u})) \subset \mathbb{R}^n \setminus H$. As $\|u - \overline{u}\|_\infty < r$, we conclude that $g(u(x))$ remains on the same side of H for every $x \in \Omega$. This contradicts the fact that $\int_\Omega g(u(x))\, dx = \int_\Omega p(x)\, dx = 0$.

Condition (6) sheds some light on the counter-example (5) of Ortega and Sánchez where the 'pathological' g rotates rapidly. Condition (6) does not allow fast rotation, as it forces $g(\overline{B}_r(v))$ to remain at one side of a hyperplane for $v \in \partial B_R(0)$.

One may ask, in first place, if rotation has the same effect as shown in [7] for higher dimensions. We shall prove that, indeed, the example by Ortega and Sánchez may be extended as follows:

Proposition 1.1 *Let $\phi : \overline{\Omega} \to \mathbb{R}$ be a non-constant eigenfunction of $-\Delta$ with Neumann boundary condition and let $p_\lambda = (\lambda\phi, 0)$. Then, problem*

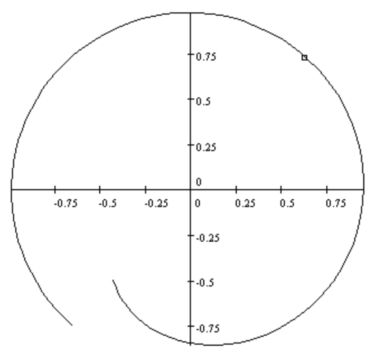
$$\begin{cases} \Delta u + g(u) = p_\lambda & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

with g as in (5) *has no solution for λ large enough.*

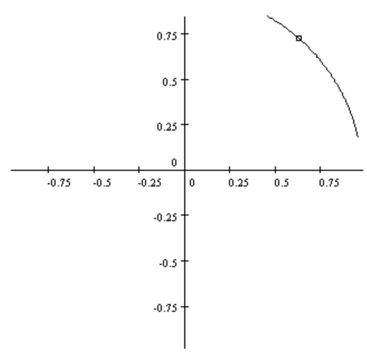
A closer look at the function (5) shows that the effect of rotation appears only when we consider the image of the whole ball $B_r(z)$ under the function g, whereas the image of a vertical strip

$$\mathcal{S}(z) := \{u \in B_r(z) : |\mathrm{Re}(u) - \mathrm{Re}(z)| < \delta\}$$

under g remains in the same half-plane for δ small enough.



$g_0(4 + t), \qquad t \in [-\pi, \pi]$



$g_0(4 + it), \;\; t \in [-\pi, \pi]$

This suggests replacing assumption (6) in Theorem 1.3 by a weaker one. We shall prove that $g(B_r(v))$ can be allowed to intersect all the hyperplanes passing through the origin, provided that, for some particular H_v, the function g maps some 'strip' in $B_r(v)$ sufficiently far away from H_v.

To make this statement precise, we need to introduce some notation. A *strip of width* 2δ in $B_r(v)$ is a set

$$\mathcal{S}(v) := \{u \in B_r(v) : |\langle u - v, \xi_v \rangle| < \delta\},$$

for some $\xi_v \in \mathbb{S}^{n-1}$. We consider the metric in Ω given by

$$d(x, y) := \inf\{\text{length}(\gamma) : \gamma \text{ is a smooth curve in } \Omega \text{ joining } x \text{ and } y\}.$$

The open ball of radius ρ for this metric will be denoted by $U_\rho(x)$, i.e.

$$U_\rho(x) := \{y \in \Omega : d(x, y) < \rho\}.$$

Further, we define

$$c(\rho) := \inf_{x \in \Omega} \text{meas}(U_\rho(x)).$$

Assume that (7) holds. For $\alpha > 1$ we choose $K > 0$ as follows: fix $\varepsilon \in [0, +\infty)$ such that

$$M_\varepsilon := \sup_{u \in \mathbb{R}^n} (|g(u)| - \varepsilon\, |u|) < \infty \tag{11}$$

and $Q\varepsilon \operatorname{diam}_d(\Omega)(1 + \alpha) < 1$, where Q is the constant in (9) and $\operatorname{diam}_d(\Omega)$ is the diameter of Ω with respect to the metric d. Next, choose $K > 0$ such that

$$K > \frac{Q(\|p\|_\infty + M_\varepsilon)}{1 - Q\varepsilon \operatorname{diam}_d(\Omega)(1 + \alpha)} > 0.$$

Our main result is the following:

Theorem 1.4 *Assume that $g \in C(\mathbb{R}^n, \mathbb{R}^n)$ satisfies (7). Let $p \in C(\overline{\Omega}, \mathbb{R}^n)$ with $\overline{p} = 0$, and $\alpha > 1$. Fix $K > 0$ as above and set $r := K \operatorname{diam}_d(\Omega)$. Assume there exists a domain $D \subset B_{\alpha r}(0)$ with the following properties:*
($\mathbf{D}_1$) *For every $v \in \partial D$ there exist a hyperplane H_v passing through the origin and a strip $\mathcal{S}(v)$ of width 2δ in $B_r(v)$ such that $g(\mathcal{S}(v)) \subset \mathbb{R}^n \setminus H_v$ and*

$$\text{dist}(g(\mathcal{S}(v)), H_v) > \kappa \text{ dist}(g(u), H_v)$$

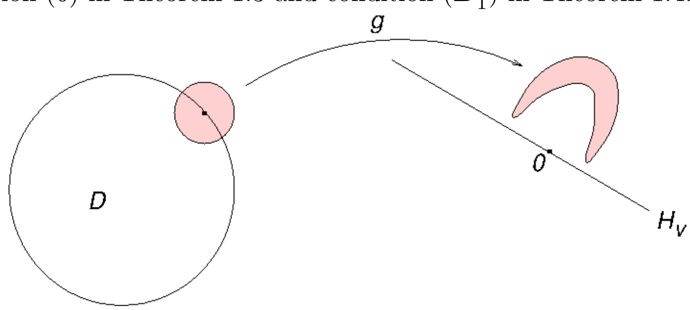
for every $u \in B_r(v)$ with $g(u) \in H_v^-$, where H_v^- denotes the closure of the connected component of $\mathbb{R}^n \setminus H_v$ not containing $g(\mathcal{S}(v))$, and $\kappa := \frac{\text{meas}(\Omega)}{c(\delta/K)} - 1$.
($\mathbf{D}_2$) $\deg(g, D, 0) \neq 0$.
Then (1) admits at least one solution u such that $\overline{u} \in D$ and $\|u - \overline{u}\|_\infty < r$.

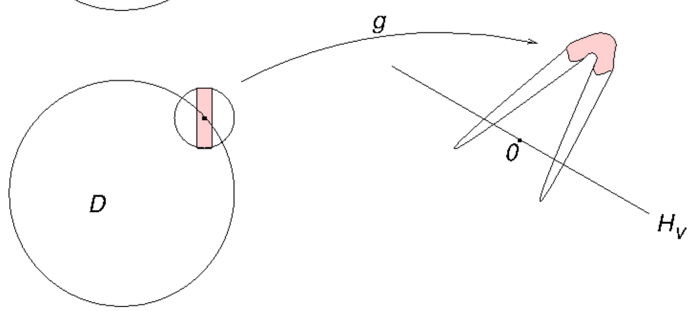
Here 'dist' denotes the euclidean distance in $\mathbb{R}^n$.

For a system of ordinary differential equations with periodic boundary conditions this result was recently established in [1]. Note that, for $N = 1$, $\text{meas}(\Omega) =$

$\mathrm{diam}_d(\Omega) = r/K$ and $c(\delta/K) = \delta/K$, so $\kappa = \frac{r}{\delta} - 1$ coincides with the constant given by Theorem 1.2 in [1], conveniently adapted for the Neumann conditions.

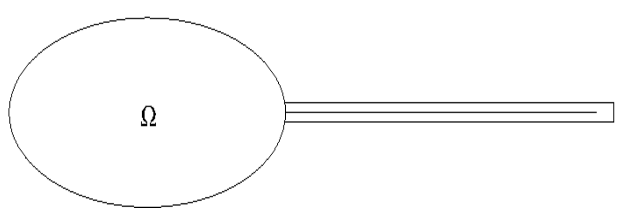
The following figures, taken from [1], illustrate the difference between condition (6) in Theorem 1.3 and condition $(\mathbf{D}_1)$ in Theorem 1.4.





Condition (6) requires that the image under g of the whole ball $\overline{B}_r(v)$ lies on one side of a hyperplane H_v through the origin, whereas condition $(\mathbf{D}_1)$ only requires the image of some strip $\mathcal{S}(v)$ to lie on one side of H_v but the image of the rest of the ball may cross the hyperplane, thus allowing for fast rotations of g. Note that $(\mathbf{D}_1)$ is trivially satisfied for any κ if (6) holds. The effect of the constant κ only appears when g rotates fast enough, that is, when $g(\overline{B}_r(v))$ intersects H_v. Then, the distance of the image of the strip to H_v is not only restricted by the rotational effect of g, as in the ODE case considered in [1], but also by the geometry of Ω, as the following example shows.

Example 1.5 *Assume that g is bounded and that condition $(\mathbf{D}_1)$ holds for some domain Ω which contains the origin, some $p \in C(\overline{\Omega}, \mathbb{R}^n)$, some $D \subset \mathbb{R}^n$ and some $\delta > 0$. Let $T_{\theta,\eta} := \{(t, y) \in \mathbb{R} \times \mathbb{R}^{N-1} : t \in [0, \theta],\ |y| \leq \eta\}$ and let Ω_η be a bounded smooth domain in $\mathbb{R}^N$ such that $\Omega \cup T_{\theta,0} \subset \Omega_\eta \subset \Omega \cup T_{\theta+1,\eta}$ for $\eta > 0$ and some θ to be established.*



Observe that the best constant Q_η for the inequality (9) *associated to the domain Ω_η is bounded from below by*

$$Q_* := \sup_{u \in C_0^2(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_\infty}{\|\Delta u\|_\infty} \leq \sup_{u \in \mathcal{A}(\Omega_\eta)\setminus\{0\}} \frac{\|\nabla u\|_\infty}{\|\Delta u\|_\infty} =: Q_\eta$$

for every $\eta > 0$, where

$$\mathcal{A}(\Omega_\eta) := \{u \in C^1(\overline{\Omega}_\eta, \mathbb{R}^n) : \|\Delta u\|_\infty < \infty, \frac{\partial u}{\partial \nu}|_{\partial \Omega_\eta} = 0\}.$$

Let $p_\eta \in C(\overline{\Omega_\eta}, \mathbb{R}^n)$ be an extension of p. Since g is bounded, we may take $\varepsilon = 0$ in (11), and $K = K_\eta > Q_\eta(\|p_\eta\|_\infty + \|g\|_\infty)$. Then,

$$\frac{\delta}{K_\eta} < \frac{\delta}{Q_*(\|p\|_\infty + \|g\|_\infty)} := d_0$$

for all $\eta > 0$. Setting θ such that $\operatorname{dist}((\theta, 0), \Omega) > d_0$, we have that the open ball $U_{\delta/K_\eta}(\theta, 0)$ for the metric d in Ω_η satisfies $U_{\delta/K_\eta}(\theta, 0) \subset T_{\theta+1,\eta}$. Therefore,

$$c\,(\delta/K_\eta) := \inf_{x \in \Omega_\eta} \operatorname{meas}(U_{\delta/K_\eta}(x)) \leq \operatorname{meas}(T_{\theta+1,\eta}) \to 0 \quad \textit{as } \eta \to 0.$$

Thus,

$$\kappa_\eta := \frac{\operatorname{meas}(\Omega_\eta)}{c\,(\delta/K_\eta)} - 1 \to \infty \quad \textit{as } \eta \to 0.$$

So condition $(\mathbf{D}_1)$ *will not hold for η sufficiently small.*

2 The proof of the main result

For the sake of completeness, let us firstly prove the existence of the constant Q introduced in (9).

By standard regularity results (see e.g. [3, Thm. 2.3.3.2]), if $u \in C(\overline{\Omega}, \mathbb{R}^n)$ is a solution of (8) then $u \in \mathcal{A}(\Omega) \subset W^{2,s}(\Omega, \mathbb{R}^n)$ for any $s < \infty$, where $\mathcal{A}(\Omega)$ is defined as in the previous section. Next, suppose that, for a sequence $(u_k) \subset \mathcal{A}(\Omega)$, $\|\nabla u_k\|_\infty > k\|\Delta u_k\|_\infty$. Let $v_k := u_k/\|\nabla u_k\|_\infty$, then $\|\Delta v_k\|_\infty \to 0$ and hence $\|\Delta v_k\|_{L^2} \to 0$. This implies that $\|\nabla v_k\|_{L^2} \to 0$ and, consequently, that $\|v_k - \overline{v}_k\|_{H^1} \to 0$. Thus $\|v_k - \overline{v}_k\|_{H^2} \to 0$ which, in turn, implies that $\|v_k - \overline{v}_k\|_{W^{1,2^*}} \to 0$. Again, we conclude that $\|v_k - \overline{v}_k\|_{W^{2,2^*}} \to 0$ and by a standard bootstrapping argument we deduce that $\|v_k - \overline{v}_k\|_{W^{2,s}} \to 0$ for some $s > N$. By the Sobolev imbedding $W^{2,s}(\Omega, \mathbb{R}^n) \hookrightarrow C^1(\overline{\Omega}, \mathbb{R}^n)$, this implies $\|\nabla v_k\|_\infty \to 0$, a contradiction.

Proof of Theorem 1.4: Consider the set

$$\mathcal{U} = \{u \in C(\overline{\Omega}, \mathbb{R}^n) : \|u - \overline{u}\|_\infty < r,\ \overline{u} \in D\}.$$

By the classical continuation theorems [4], it suffices to prove that problem (8) has no solution on $\partial\mathcal{U}$ for $\lambda \in (0,1]$.

Assume that u satisfies (8) for some $\lambda \in (0,1]$. Then

$$\|\nabla u\|_\infty \le Q\|\Delta u\|_\infty \le Q(\|p\|_\infty + \varepsilon\|u\|_\infty + M_\varepsilon),$$

where $\varepsilon > 0$ is the number such that $Q\varepsilon\,\mathrm{diam}_d(\Omega)(1+\alpha) < 1$ chosen to define K, and M_ε is given by (11). Thus,

$$\|\nabla u\|_\infty \le Q\|\Delta u\|_\infty \le Q\left(\|p\|_\infty + M_\varepsilon + \varepsilon\left[|\overline{u}| + \mathrm{diam}_d(\Omega)\|\nabla u\|_\infty\right]\right). \qquad (12)$$

As $D \subset B_{\alpha r}(0)$, it follows that $|\overline{u}| < \alpha K \mathrm{diam}_d(\Omega)$. We claim that

$$\|\nabla u\|_\infty < K \qquad \text{and} \qquad \|u - \overline{u}\|_\infty < r. \qquad (13)$$

Indeed, if $\|\nabla u\|_\infty \ge K$, inequality (12) would yield

$$K\left[1 - Q\varepsilon\,\mathrm{diam}_d(\Omega)(1+\alpha)\right] \le Q(\|p\|_\infty + M_\varepsilon),$$

contradicting our choice of K. Thus, $\|\nabla u\|_\infty < K$, which implies $\|u - \overline{u}\|_\infty < r$. It remains to prove that $\overline{u} \notin \partial D$.

Taking w_v as the unit normal vector to H_v such that $\langle g(v), w_v\rangle > 0$, it is straightforward to check that condition $(\mathbf{D}_1)$ is equivalent to
$(\mathbf{D}_1')$ For each $v \in \partial D$ there exist a vector $w_v \in \mathbb{S}^{n-1}$ and a strip $\mathcal{S}(v)$ of width 2δ in $B_r(v)$ such that

$$\inf_{y\in\mathcal{S}(v)} \langle g(y), w_v\rangle + \left(\frac{\mathrm{meas}(\Omega)}{c(\delta/K)} - 1\right)\langle g(u), w_v\rangle > 0 \qquad (14)$$

for every $u \in B_r(v)$ such that $\langle g(u), w_v\rangle \le 0$.

Next, arguing by contradiction, suppose that $\overline{u} \in \partial D$ and take $w_{\overline{u}} \in \mathbb{S}^{n-1}$ and the strip $\mathcal{S}(\overline{u}) = \{u \in B_r(\overline{u}) : |\langle u - \overline{u}, \xi_{\overline{u}}\rangle| < \delta\}$ with $\xi_{\overline{u}} \in \mathbb{S}^{n-1}$ such that (14) holds for $v = \overline{u}$. As u solves (8), we have that

$$0 = \int_\Omega \langle g(u(x)), w_{\overline{u}}\rangle\, dx = \int_\Omega \langle g(u(x)) - Tw_{\overline{u}}, w_{\overline{u}}\rangle\, dx + T\ \mathrm{meas}(\Omega),$$

where

$$T := \inf_{x\in\Omega} \langle g(u(x)), w_{\overline{u}}\rangle.$$

Hence, $T \le 0$.

Define $\varphi(u) := \langle u, \xi_{\overline{u}}\rangle$. From the mean value theorem for vector integrals we deduce that $\overline{u} \in \mathrm{co}(\mathrm{Im}(u))$. Thus, $\varphi(\overline{u}) \in \varphi(\mathrm{Im}(u))$. Consequently, we may fix $\overline{x} \in \Omega$ such that $\varphi(u(\overline{x})) = \varphi(\overline{u})$, and from (13) we obtain

$$|\varphi(u(x)) - \varphi(\overline{u})| \le |u(x) - u(\overline{x})| \le K\, d(x, \overline{x}).$$

This implies that $u(x) \in \mathcal{S}(\overline{u})$ for $d(x,\overline{x}) < \frac{\delta}{K}$. Thus, if

$$A := \{x \in \Omega : u(x) \in \mathcal{S}(\overline{u})\},$$

then $U_{\delta/K}(\overline{x}) \subset A$, and hence $\mathrm{meas}(A) \geq c(\delta/K)$. Moreover, as $\overline{\Omega}$ is compact, we may choose $x_0 \in \overline{\Omega}$ such that $\langle g(u(x_0)), w_{\overline{u}}\rangle = T \leq 0$. Then,

$$\begin{aligned} 0 &\geq \int_A \langle g(u(x)) - Tw_{\overline{u}}, w_{\overline{u}}\rangle \, dx + T\mathrm{meas}(\Omega) \\ &\geq c(\delta/K) \inf_{v\in\mathcal{S}(\overline{u})} \langle g(v), w_{\overline{u}}\rangle + T\left(\mathrm{meas}(\Omega) - c(\delta/K)\right) \\ &= c(\delta/K)\left[\inf_{v\in\mathcal{S}(\overline{u})} \langle g(u), w_{\overline{u}}\rangle + \left(\frac{\mathrm{meas}(\Omega)}{c(\delta/K)} - 1\right)\langle g(u(x_0)), w_{\overline{u}}\rangle\right], \end{aligned}$$

which contradicts (14). $\square$

3 The proof of the nonexistence result

The following lemma will be used to prove Proposition 1.1.

Lemma 3.1 *Let $U \subset \mathbb{R}^N$ be a smooth bounded domain, $\Gamma \in C^1(\mathbb{R},\mathbb{R})$, $h_k \in C^1(\overline{U},\mathbb{R})$, $\varphi, \omega_k \in C^2(\overline{U},\mathbb{R})$, $A_k, \lambda_k \in \mathbb{R}$ and $\alpha > 1$ be such that*

$$|\nabla\varphi(x)| \geq \frac{1}{\alpha} \qquad \textit{for all } x \in \overline{U},$$

$$\|\Gamma\|_{C^1}, \|h_k\|_{C^1}, \|\varphi\|_{C^2}, \|\omega_k\|_{C^1} \leq \alpha, \|\omega_k''\|_\infty \leq \alpha\lambda_k.$$

Assume that $\lambda_k \to +\infty$. Then

$$\lim_{k\to\infty} \int_U h_k(x)\Gamma'(\lambda_k\varphi(x) + \omega_k(x) + A_k)\, dx = 0.$$

<u>*Proof*</u>: We consider two cases.

Case 1: $N = 1$.

Let $U = (a,b)$. Since $|\varphi'(t)| \geq \frac{1}{\alpha}$ for all $t \in [a,b]$ and $\|\omega_k'\|_{C^0} \leq \alpha$, there exists $\lambda_* > 0$, independent of k, such that $\left|\varphi'(t) + \frac{1}{\lambda}\omega_k'(t)\right| \geq \frac{1}{2\alpha}$ for all $t \in [a,b]$ and $\lambda \in [\lambda_*, \infty)$. In particular, the function

$$f_k(t) := \frac{h_k(t)}{\varphi'(t) + \frac{1}{\lambda_k}\omega_k'(t)}$$

is well defined for $\lambda_k \in [\lambda_*, \infty)$. Since

$$f_k'(t) := \frac{h_k'(t)\left(\varphi'(t) + \frac{1}{\lambda_k}\omega_k'(t)\right) - h_k(t)\left(\varphi''(t) + \frac{1}{\lambda_k}\omega_k''(t)\right)}{\left(\varphi'(t) + \frac{1}{\lambda_k}\omega_k'(t)\right)^2},$$

we have that $\|f_k\|_\infty \leq 2\alpha^2$, $\|f_k'\|_\infty \leq 16\alpha^4$. Integrating by parts we obtain

$$\int_a^b h_k(t)\Gamma'(\lambda_k\varphi(t) + \omega_k(t) + A_k)\, dt = \frac{1}{\lambda_k}\int_a^b f_k(t)\frac{d}{dt}[\Gamma(\lambda_k\varphi(t) + \omega_k(t) + A_k)]\, dt$$

$$= \frac{1}{\lambda_k}\left[f_k(\cdot)\Gamma(\lambda_k\varphi(\cdot)+\omega_k(\cdot)+A_k)\Big|_a^b - \int_a^b f_k'(t)\Gamma(\lambda_k\varphi(t)+\omega_k(t)+A_k)\,dt\right].$$

Hence, if $\lambda_k \in [\lambda_*, \infty)$,

$$\left|\int_a^b h_k(t)\Gamma'(\lambda_k\varphi(t)+\omega_k(t)+A_k)\,dt\right| \leq \frac{16\alpha^5(b-a+1)}{\lambda_k}.$$

As $\lambda_k \to \infty$, the result follows.

Case 2: $N > 1$.

Let $U_1, \ldots, U_N$ be open subsets such that $\overline{U} = \cup_{i=1}^N \overline{U}_i$ and

$$\left|\frac{\partial\varphi}{\partial x_i}(x)\right| \geq \frac{1}{\sqrt{N}\alpha} \qquad \text{for all } x \in \overline{U}_i.$$

Write $x = (t, y) \in \mathbb{R} \times \mathbb{R}^{N-1}$, and set $J_{1,y} := \{t \in \mathbb{R} : (t, y) \in U_1\}$. Then, for each $y \in \mathbb{R}^{N-1}$, we may apply the case $N = 1$ to conclude that

$$\int_{J_{1,y}} h_k(t,y)\Gamma'(\lambda_k\varphi(t,y)+\omega_k(t,y)+A_k)\,dt \to 0.$$

Since U_1 is bounded and $\{h_k\Gamma'(\lambda_k\varphi+\omega_k+A_k)\}$ is uniformly bounded in U, Fubini's theorem and the dominated convergence theorem yield

$$\lim_{k\to\infty}\int_{U_1} h_k(x)\Gamma'(\lambda_k\varphi(x)+\omega_k(x)+A_k)\,dx = 0.$$

Similarly for $U_2, \ldots, U_N$. Thus, the result follows. □

Proof of Proposition 1.1 : Arguing by contradiction, assume there is a sequence $\lambda_k \to \infty$ such that problem (10) has a solution z_k. For convenience, from now on we shall write p_k instead of p_{λ_k}. Define $w_k = z_k + \frac{1}{\mu}p_k$, where μ is the eigenvalue associated to ϕ. Then

$$\Delta w_k = \Delta z_k + \frac{1}{\mu}\Delta p_k = p_k - g(z_k) - p_k,$$

that is to say,

$$\Delta w_k + g_0(w_k - \frac{1}{\mu}p_k) = \gamma. \tag{15}$$

Next, observe that

$$\int_\Omega \Delta w_k = \int_{\partial\Omega}\frac{\partial w_k}{\partial\nu} = \int_{\partial\Omega}\frac{\partial z_k}{\partial\nu} + \frac{1}{\mu}\int_{\partial\Omega}\frac{\partial p_k}{\partial\nu} = 0,$$

so integrating (15) yields

$$\int_\Omega g_0(w_k - \frac{1}{\mu}p_k) = \gamma\,\text{meas}(\Omega). \tag{16}$$

Fix a positive $\varepsilon < \gamma \mathrm{meas}(\Omega)$. Since the critical set $\mathcal{C}$ of ϕ is a compact subset of $\overline{\Omega}$ and has zero measure (see Lemma 3.2 below), we may fix a smooth domain U_ε with $\overline{U}_\varepsilon \subset \Omega$ such that $\overline{U}_\varepsilon \cap \mathcal{C} = \emptyset$ and

$$\left| \int_{\Omega \backslash U_\varepsilon} g_0(w_k - \frac{1}{\mu} p_k) \right| \leq \ \mathrm{meas}(\Omega \backslash U_\varepsilon) < \varepsilon.$$

Writing $w_k = x_k + i y_k$ we compute

$$\begin{aligned} g_0(w_k - \frac{1}{\mu} p_k) &= \frac{w_k - \frac{1}{\mu} p_k}{\sqrt{1 + |w_k - \frac{1}{\mu} p_k|^2}} e^{i \mathrm{Re}(w_k - \frac{1}{\mu} p_k)} \\ &= \frac{(x_k - \frac{\lambda_k}{\mu} \phi) \cos(x_k - \frac{\lambda_k}{\mu} \phi) - y_k \sin(x_k - \frac{\lambda_k}{\mu} \phi)}{\sqrt{1 + |w_k - \frac{1}{\mu} p_k|^2}} \\ &\quad + i \frac{y_k \cos(x_k - \frac{\lambda_k}{\mu} \phi) + (x_k - \frac{\lambda_k}{\mu} \phi) \sin(x_k - \frac{\lambda_k}{\mu} \phi)}{\sqrt{1 + |w_k - \frac{1}{\mu} p_k|^2}}. \end{aligned}$$

Next we write the previous equality as $g_0 = g_0^1 + g_0^2 + i(g_0^3 + g_0^4)$ and apply Lemma 3.1 to each one of these four summands. For example, for the first one we set

$$h_k := \frac{x_k - \frac{\lambda_k}{\mu} \phi}{\sqrt{1 + |w_k - \frac{1}{\mu} p_k|^2}}, \quad \Gamma := \sin, \quad \varphi := \frac{1}{\mu} \phi$$

$$\omega_k := x_k - \overline{x_k}, \quad A_k := \overline{x_k}.$$

As $\|\Delta w_k\|_\infty \leq |\gamma| + 1$, a uniform bound (i. e. independent of k) for ω_k in $C^1(\overline{U_\varepsilon}, \mathbb{R})$ follows from the standard Sobolev estimates. However, Lemma 3.1 cannot be applied yet since $\|h_k\|_{C^1(\overline{U_\varepsilon}, \mathbb{R})}$ or $\|\omega_k / \lambda_k\|_{C^2(\overline{U_\varepsilon}, \mathbb{R})}$ might not be uniformly bounded. In order to overcome this difficulty, take a subsequence if necessary to define

$$\rho := \lim_{k \to \infty} \frac{\mu \overline{x_k}}{\lambda_k}.$$

Suppose firstly that $|\rho| < \infty$. As $\|w_k - \overline{w_k}\|_\infty$ is bounded, $\frac{\mu x_k}{\lambda_k}$ converges uniformly to ρ.

For each $\delta > 0$ there exists a constant $c_\delta > 0$ such that

$$|w_k - \frac{1}{\mu} p_k| \geq |\mathrm{Re}(w_k - \frac{1}{\mu} p_k)| = \frac{\lambda_k}{\mu} \left| \frac{\mu x_k}{\lambda_k} - \phi \right| \geq c_\delta \lambda_k$$

on $\Omega_\delta := \phi^{-1}([\rho - \delta, \rho + \delta]^c)$. Moreover, as $\|\nabla w_k\|_\infty$ is bounded, it follows that $\|\nabla (w_k - \frac{1}{\mu} p_k)\|_\infty = O(\lambda_k)$. Thus, if we set

$$\theta(x, y) := \frac{x}{\sqrt{1 + x^2 + y^2}},$$

then $|\nabla\theta(w_k - \frac{1}{\mu}p_k)| \leq \frac{1}{\sqrt{1+|w_k-\frac{1}{\mu}p_k|^2}} \leq \frac{1}{c_\delta \lambda_k}$ on Ω_δ. Using the chain rule, we conclude that $\theta(w_k - \frac{1}{\mu}p_k)$ is bounded in $C^1(\overline{\Omega}_\delta, \mathbb{R})$.

The same conclusion holds for

$$\chi(x,y) := \frac{y}{\sqrt{1+x^2+y^2}};$$

in particular, $\|\nabla g_0(w_k - \frac{1}{\mu}p_k)\|_\infty = O(\lambda_k)$. Thus, using (15), the interior Schauder estimates provide a uniform C^2 bound for $\frac{\omega_k}{\lambda_k}$ over $\overline{U_\varepsilon \cap \Omega_\delta}$.

As $\Omega = \phi^{-1}(\rho) \cup \bigcup_{k\in\mathbb{N}} \Omega_{1/k}$ and $\text{meas}(\phi^{-1}(\rho)) = 0$, we may fix $\delta > 0$ such that $\text{meas}(\Omega \setminus \Omega_\delta)$ is arbitrarily small.

Taking α large enough, Lemma 3.1 implies that

$$\lim_{k\to\infty} \int_{U_\varepsilon \cap \Omega_\delta} g_0^1(w_k - \frac{1}{\mu}p_k) = 0$$

and hence

$$\lim_{k\to\infty} \int_{U_\varepsilon} g_0^1(w_k - \frac{1}{\mu}p_k) = 0.$$

If we suppose, on the contrary, that $\rho = \pm\infty$, then it is immediately seen that for some $c > 0$

$$|w_k - \frac{1}{\mu}p_k| \geq c\lambda_k$$

on Ω, and the conclusion follows.

The procedure is similar for the other summands. We conclude that

$$\limsup_{k\to\infty} \left| \int_\Omega g_0(w_k - \frac{1}{\mu}p_k) \right| \leq \varepsilon,$$

which contradicts (16). □

Lemma 3.2 *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be analytic with $\psi \not\equiv 0$. Then $meas\{\psi = 0\} = 0$.*

<u>*Proof*</u>: We proceed by induction. The case $n = 1$ is trivial. For each $x \in \mathbb{R}^{n-1}$ the functions $\psi^t(x) = \psi_x(t) = \psi(t,x)$ are analytic.

Fix t such that $\psi^t \not\equiv 0$. From the inductive hypothesis,

$$meas(\{x/\psi^t(x) = 0\}) = 0$$

and also

$$meas(\{x/\psi_x \equiv 0\}) = 0.$$

On the other hand, $meas(\{t/\psi_x(t) = 0\}) = 0$ for every x such that $\psi_x \not\equiv 0$. Thus, if we define

$$NZ = \{(t,x)/\psi(t,x) = 0 \text{ and } \psi_x \not\equiv 0\},$$

then by Fubini's Theorem we deduce that $meas(NZ) = 0$. We conclude that

$$\{\psi = 0\} = \mathbb{R} \times \{x/\psi_x \equiv 0\} \cup NZ$$

has zero measure. □

References

[1] P. Amster and M. Clapp, *Periodic solutions of resonant systems with rapidly rotating nonlinearities* Discrete and Continuous Dynamical Systems, Series A **31** No. 2 (2011), 373–383.

[2] P. Amster and M. Maurette, *An Elliptic Singular System with Nonlocal Boundary Conditions*. Submitted.

[3] P. Grisvard, Elliptic problems in nonsmooth domains. Pitman, Boston, 1985.

[4] J. Mawhin, Topological degree methods in nonlinear boundary value problems, volume 40 of CBMS Regional Conference Series in Mathematics. American Mathematical Society, Providence, R.I., 1979. Expository lectures from the CBMS Regional Conference held at Harvey Mudd College, Claremont, Calif. (1977), 9–15.

[5] L. Nirenberg, *Generalized degree and nonlinear problems*, in "Contributions to Nonlinear Functional Analysis" (E. H. Zarantonello ed.), Academic Press New York, (1971), 1–9.

[6] R. Ortega, *A counterexample for the damped pendulum equation*, Acad. Roy. Belg. Bull. Cl. Sci., **73** (1987), 405–409.

[7] R. Ortega and L. Sánchez, *Periodic solutions of forced oscillators with several degrees of freedom*, Bull. London Math. Soc., **34** (2002), 308–318.

[8] R. Ortega, E. Serra and M. Tarallo, *Non-continuation of the periodic oscillations of a forced pendulum in the presence of friction*, Proc. Amer. Math. Soc., **128** (2000), 2659–2665.

[9] D. Ruiz and J. R. Ward Jr., *Some notes on periodic systems with linear part at resonance*, Discrete and Continuous Dynamical Systems, **11** (2004), 337–350.

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